ARC49

C2 - Actuarial Statistics I

Extrapolating co-linear payment year trends for development triangle GLMs

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Learning Objectives

1. Maximum number of parameters for a multiplicative triangle GLM that includes exposure, development, and payment periods
2. Structure of incremental trend model
3. Interpretation of fitted parameters: cannot measure absolute value of trends in single dimension of analysis
4. Extrapolation of future payment period trends: need dynamic adjustment to avoid biased bootstrap

Outline (1/2)

A. Basic model structure
   i. Multiplicative model with discrete parameters for each exposure, development, and payment period
   ii. Slack factors that reduce the effective dimensions of the space of modeled triangles
   iii. Unique parameterization by fixing selected parameter values

B. Trend model (log-scale)
   i. Incremental trends
   ii. Parameters values depend on reference periods and they are correlated across dimensions of analysis
   iii. Co-linear vs. independent dimensions of analysis

Outline (2/2)

C. Offset invariant extrapolation
   i. Intuition: fitted triangle values do not depend on specific parameterization; looking for an extrapolation method that has the same property
   ii. Dynamically mixing the fitted trends (weights adding to one) does the trick; each future payment period trend can be extrapolated on its own; can be combined with additional constant adjustment
   iii. Method replicates bootstrapping results for model without payment period parameters
   iv. Unlike static extrapolation the method avoids biased bootstrap

Data from Taylor & Ashe (1983)

Incremental Input Values

Period	Dev	1	2	3	4	5	6	7	8	9	10
Exp											
1		357,848	766,940	610,542	482,940	527,326	574,398	146,342	139,950	227,229	67,948
2		352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
3		290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
4		310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
5		443,160	693,190	991,983	769,488	504,851	470,639				
6		396,132	937,085	847,498	805,037	705,960					
7		440,832	847,631	1,131,398	1,063,269						
8		359,480	1,061,648	1,443,370							
9		376,686	986,608								
10		344,014									

Data from Taylor & Ashe (1983)

Fitted Incremental Values Maximal model with 27 parameters

Period	Dev	1	2	3	4	5	6	7	8	9	10
Exp											
1		357,848	719,008	617,974	666,748	467,856	283,583	316,627	150,625	253,245	67,948
2		400,050	842,014	896,226	1,195,112	559,160	483,086	308,404	256,003	399,030	
3		325,082	847,343	1,114,697	991,117	660,957	326,505	363,715	279,899		
4		318,924	1,027,430	901,212	1,142,130	435,503	375,392	387,678			
5		383,148	823,017	1,028,973	745,625	496,104	396,443				
6		344,219	1,053,896	753,391	952,607	587,599					
7		464,785	813,661	1,014,946	1,189,738						
8		377,544	1,153,281	1,333,674							
9		355,772	1,007,522								
10		344,014									

Data from Taylor & Ashe (1983)

Fitted Incremental Values Last five diagonals, 9 exposure trends, 9 development trends, 1 payment trend

Period	Dev	1	2	3	4	5	6	7	8	9	10
Exp											
1							320,206	291,147	192,373	284,193	67,948
2						576,713	414,724	377,088	249,157	368,082	88,005
3					1,100,118	567,084	407,799	370,792	244,997	361,936	86,535
4				943,895	1,001,037	516,010	371,071	337,397	222,932	329,339	78,742
5			749,730	893,370	947,453	488,389	351,209	319,337	210,999	311,710	74,527
6		339,982	816,155	972,520	1,031,396	531,659	382,325	347,629	229,693	339,327	81,130
7		374,741	899,597	1,071,949	1,136,844	586,015	421,413	383,170	253,176	374,019	89,424
8		457,508	1,098,286	1,308,704	1,387,932	715,445	514,488	467,799	309,094	456,626	109,175
9		400,900	962,394	1,146,777	1,216,202	626,922	450,830	409,918	270,849	400,127	95,667
10		344,014	825,835	984,055	1,043,629	537,965	386,860	351,752	232,417	343,351	82,092

A. Basic model structure

i. Multiplicative model with discrete parameters for each exposure, development, and payment period

$$\mu_{ij} = a_i \cdot b_j \cdot c_{i+j-1},$$

where $i, j = 1, 2, \ldots, n$ with $i + j \leq n + 1$, and $a_i, b_j, c_{i+j-1} > 0$.

A. Basic model structure

ii. Slack factors that reduce the effective dimensions of the space of modeled triangles

$$\mu_{ij} = a_i \cdot b_j \cdot c_{i+j-1} = a_i' \cdot b_j' \cdot c_{i+j-1}',$$

$$a_i' = \frac{x}{z^i} a_i, \quad b_j' = \frac{y}{z^j} b_j, \quad c_k' = \frac{z^{k+1}}{x \cdot y} c_k,$$

where $x, y, z > 0$.

A. Basic model structure

iii. WLOG we may chose reference levels r, s, t with $r + s \neq t + 1$ such that $a'_r = b'_s = c'_t = 1$.

Proof: given general parameterization, use

$$z = (a_r \cdot b_s \cdot c_t)^{1/(r+s-t-1)}, \quad x = \frac{z^r}{a_r}, \quad y = \frac{z^s}{b_s}.$$

This implies that a triangle GLM has at most $3n - 3$ parameters (for $n \times n$ triangle).

B. Trend model (log-scale)

i. Using the log link function and switching to incremental trend parameters we get the following

$$\eta_{ij} = -\sum_{\ell=i}^{r-1}\alpha_\ell - \sum_{\ell=j}^{s-1}\beta_\ell - \sum_{\ell=i+j-1}^{t-1}\gamma_\ell$$

$$+\sum_{\ell=r}^{i-1}\alpha_\ell + \sum_{\ell=s}^{j-1}\beta_\ell + \sum_{\ell=t}^{i+j-2}\gamma_\ell$$

where $\alpha_\ell, \beta_\ell, \gamma_\ell$ are the incremental trend parameters, with $\ell = 1, 2, \dots, n-1$.

B. Trend model (log-scale)

ii. Parameter values as a function of reference level

$r = 4, s = 5, t = 5$

ℓ	1	2	3	4	5	6	7	8	9
α_ℓ	-4.1	-4.5	-4.5	-4.5	-4.4	-4.3	-4.3	-4.7	-4.6
β_ℓ	-3.5	-4.4	-4.4	-5.0	-4.8	-4.5	-4.8	-4.1	-5.9
γ_ℓ	4.2	4.3	4.5	4.7	4.3	4.6	4.1	4.6	4.6

$r = 4, s = 5, t = 6$

ℓ	1	2	3	4	5	6	7	8	9
α_ℓ	-6.2	-6.6	-6.6	-6.6	-6.5	-6.5	-6.4	-6.8	-6.7
β_ℓ	-5.7	-6.5	-6.5	-7.2	-6.9	-6.7	-7.0	-6.2	-8.0
γ_ℓ	6.4	6.4	6.6	6.8	6.4	6.8	6.2	6.8	6.7

All parameter values are shifted by ±2.139; fitted data values unchanged.

Data: Taylor and Ashe (1983), ODP model ($V(\mu) = \phi\mu$) fitted to full triangle

B. Trend model (log-scale)

iii. Co-linear vs. independent dimensions of analysis

Co-linear

$$\eta_{ij} = -\sum_{\ell=i}^{r-1} \alpha_\ell - \sum_{\ell=j}^{s-1} \beta_\ell - \sum_{\ell=i+j-1}^{t-1} \gamma_\ell + \sum_{\ell=r}^{i-1} \alpha_\ell + \sum_{\ell=s}^{j-1} \beta_\ell + \sum_{\ell=t}^{i+j-2} \gamma_\ell$$

$r + s \neq t + 1$

$k = i + j - 1$ (implicit)

No constant offset

$3(n-1)$ parameters

Independent

$$\eta_{ijk} = \kappa - \sum_{\ell=i}^{r-1} \alpha_\ell - \sum_{\ell=j}^{s-1} \beta_\ell - \sum_{\ell=k}^{t-1} \gamma_\ell + \sum_{\ell=r}^{i-1} \alpha_\ell + \sum_{\ell=s}^{j-1} \beta_\ell + \sum_{\ell=t}^{k-1} \gamma_\ell$$

All combinations of r, s, t allowed

k (independent index)

Constant offset κ

$1 + 3(n-1)$ parameters

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iii. Co-linear vs. independent dimensions of analysis

Co-linear

$$\eta_{rs} = -\sum_{\ell=r+s-1}^{t-1} \gamma_\ell + \sum_{\ell=t}^{r+s-2} \gamma_\ell$$

$$\eta_{r(t-r+1)} = -\sum_{\ell=t-r+1}^{s-1} \beta_\ell + \sum_{\ell=s}^{t-r} \beta_\ell$$

$$\eta_{(t-s+1)s} = -\sum_{\ell=t-s+1}^{r-1} \alpha_\ell + \sum_{\ell=r}^{t-s} \alpha_\ell$$

Remember $r + s \neq t + 1$

Independent

$$\eta_{rst} = \kappa$$

Trend parameters can be interpreted as incremental offsets relative to base cell. The only parameter that changes when different reference levels are chosen is the κ parameter.

C. Offset invariant extrapolation

i. Intuition:
   - Goodness of fit measure of model (i.e. likelihood) only depends on fitted values, not the specific parameterization
   - Want extrapolation method that is invariant under changes in reference levels

C. Offset invariant extrapolation

ii. Dynamic mixing – the mechanics

$$\gamma_k = \delta_k + \sum_{\ell=1}^{n-1} \omega_{k\ell} \cdot \gamma_\ell \,,$$

where $k = n, \dots, 2n-2$, and $\sum_{\ell=1}^{n-1} \omega_{k\ell} = 1$. Ensuring that

$$\eta_{ij} = \quad -\sum_{\ell=i}^{r-1} \alpha_\ell - \sum_{\ell=j}^{s-1} \beta_\ell - \sum_{\ell=i+j-1}^{t-1} \gamma_\ell$$
$$+\sum_{\ell=r}^{i-1} \alpha_\ell + \sum_{\ell=s}^{j-1} \beta_\ell + \sum_{\ell=t}^{i+j-2} \gamma_\ell$$

now also works for $i + j > n + 1$, thus allowing us to square the triangle.

C. Offset invariant extrapolation

ii. Dynamic mixing – why does it work

- While we cannot rely on the absolute value of the fitted payment period trends, using a mixture with weights summing to one ensures that the extrapolated parameters follow any shifts experienced by the fitted parameters. The extrapolated values are therefore independent of the reference levels chosen.
- The method is flexible and allows to express actuarial judgment such as “the next two years should see a payment year trend similar to the most recent observed; beyond that we expect payment year trends to taper towards the long term average.”
- Based on exogenous information we can also model effects such as “over the next five years we expect to see payment period trends that are 1% below the average trend observed in the triangle.”

C. Offset invariant extrapolation

iii. Replicating model with no payment period dimension

- In practice we do not use the maximal model introduced in B.i. Instead we try to reduce the number of parameters by grouping together selected trends. This is the GLM equivalent of the Barnett and Zehnwirth PTF model.
- By allowing for distinct trend parameters for each exposure and development period, while assuming that all payment period trends are the same, we can replicate the results of the constant offset model that ignores the payment period dimension of analysis.
- For example, performing a 50,000 iteration bootstrap of the last five diagonals of data from Taylor and Ashe (1983) produces identical results: standard error of the reserve outcome of 22.45%, moderate bias (over-projection) of 1.6%, estimated reserve of 18.9M.

C. Offset invariant extrapolation

iv. Comparison with static extrapolation

- While the method presented here depends on exogenous assumptions, it is consistent with the general framework for using bootstrapping to derive a distribution of reserve outcomes.
- Using static extrapolation (future payment period parameters are the same for all bootstrap iterations) seems to leave out consideration of parameter uncertainty. Moreover, a bootstrap with static extrapolation introduces significant bias and runs with 50,000 iterations do not produce a robust estimate of the standard error of reserve outcomes.
- For example, 50,000 iteration bootstraps for the same model mentioned on the last slide result in significant bias (over-projection) of about 18%, while estimates of the standard error are all over the place (e.g 87% or 305%).

Review

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Review

B. Trend model (log-scale)

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$$\eta_{ij} = -\sum_{\ell=i}^{r-1}\alpha_\ell - \sum_{\ell=j}^{s-1}\beta_\ell - \sum_{\ell=i+j-1}^{t-1}\gamma_\ell + \sum_{\ell=r}^{i-1}\alpha_\ell + \sum_{\ell=s}^{j-1}\beta_\ell + \sum_{\ell=t}^{i+j-2}\gamma_\ell$$

$r + s \neq t + 1$

$k = i + j - 1$ (implicit)

No constant offset

$3(n-1)$ parameters

Independent

$$\eta_{ijk} = \kappa - \sum_{\ell=i}^{r-1}\alpha_\ell - \sum_{\ell=j}^{s-1}\beta_\ell - \sum_{\ell=k}^{t-1}\gamma_\ell + \sum_{\ell=r}^{i-1}\alpha_\ell + \sum_{\ell=s}^{j-1}\beta_\ell + \sum_{\ell=t}^{k-1}\gamma_\ell$$

All combinations of r, s, t allowed

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γ_ℓ	4.2	4.3	4.5	4.7	4.3	4.6	4.1	4.6	4.6

$r = 4, s = 5, t = 6$

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Contact Information

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- P.S.: Do you have any suggestions for journal to publish this type of material (other than "Variance")?